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著者	Umezu Kenichiro, Taira Kazuaki
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# GROWING-UP POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

KENICHIRO UMEZU AND KAZUAKI TAIRA

Maebashi Institute of Technology, Maebashi 371-0816, Japan,  
Institute of Mathematics, University of Tsukuba, Tsukuba 305-8571, Japan

ABSTRACT. This paper is devoted to the study of the existence, uniqueness and asymptotic behavior of positive solutions of a class of *degenerate* boundary value problems for semilinear second-order elliptic differential operators which originates from the so-called Yamabe problem in Riemannian geometry. Our approach is based on the super-sub-solution method adapted to the degenerate case.

## 1. INTRODUCTION AND RESULTS

Let  $D$  be a bounded domain of Euclidean space  $\mathbf{R}^N$ ,  $N \geq 2$ , with smooth boundary  $\partial D$ ; its closure  $\overline{D} = D \cup \partial D$  is an  $N$ -dimensional, compact smooth manifold with boundary. This paper is devoted to the study of the existence and uniqueness of positive solutions of the following semilinear elliptic boundary value problem:

$$\begin{cases} -\Delta u = \lambda u - h(x) u^p & \text{in } D, \\ Bu := a(x') \frac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))u = 0 & \text{on } \partial D. \end{cases} \quad (*)_\lambda$$

Here:

- (1)  $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \dots + \partial^2/\partial x_N^2$  is the usual Laplacian.
- (2)  $\lambda$  is a positive parameter.
- (3)  $h(x)$  is a real-valued function on the closure  $\overline{D}$ .
- (4)  $p > 1$ .
- (5)  $\mathbf{n} = (n_1, n_2, \dots, n_N)$  is the unit exterior normal to the boundary  $\partial D$ .
- (6)  $a(x')$  is a real-valued function on the boundary  $\partial D$ .

A function  $u(x) \in C^2(\overline{D})$  is called a *positive solution* of problem  $(*)_\lambda$  if it satisfies problem  $(*)_\lambda$  and is strictly positive everywhere in  $D$ .

It is worth pointing out here that the equation  $-\Delta u - \lambda u + h(x) u^p = 0$  originates from the so-called Yamabe problem which is a basic problem in Riemannian geometry if we take  $p = (N + 2)/(N - 2) > 1$  for  $N \geq 3$  (see [7], [8]).

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Our fundamental conditions on the function  $h(x)$  are the following:

$$h(x) \in C^\theta(\overline{D}), \quad 0 < \theta < 1; \quad (\text{H.1})$$

$$h(x) \geq 0 \quad \text{on } \overline{D}. \quad (\text{H.2})$$

We remark that Ouyang [9] and Korman-Ouyang [6] studied the case where the function  $h(x)$  may change sign in  $D$ .

On the other hand, our boundary condition  $B$  is a linear combination of the Dirichlet and Neumann conditions. It is easy to see that the boundary condition  $B$  is non-degenerate (or coercive) if and only if either  $a(x') \neq 0$  on  $\partial D$  or  $a(x') \equiv 0$  on  $\partial D$ . Ouyang [8] and del Pino [3] studied the Dirichlet and Neumann cases, while Fraile et al. [4] studied the general non-degenerate case. For further studies of semilinear elliptic problems, we refer to Alama-Tarantello [1], Amann [2], Gámez [5] and Pao [10].

In this paper we study problem  $(*)_\lambda$  in the *degenerate* case; more precisely, our fundamental condition on the function  $a(x')$  is the following:

$$0 \leq a(x') \leq 1 \quad \text{on } \partial D. \quad (\text{H.3})$$

Remark that the so-called Lopatinskii-Shapiro complementary condition is violated at the points  $x' \in \partial D$  where  $a(x') = 0$ .

In order to formulate our results, let  $\lambda_1$  be the first eigenvalue of the linearized eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } D, \\ B\varphi = 0 & \text{on } \partial D. \end{cases} \quad (1.1)$$

It is known (see [11, Theorem 1]) that the first eigenvalue  $\lambda_1$  is non-negative and simple and further that its associated eigenfunction  $\varphi_1(x)$  can be chosen to be positive everywhere in  $D$ . By Green's formula, it is easy to see that a necessary condition on the parameter  $\lambda$  for the existence of positive solutions of problem  $(*)_\lambda$  is that  $\lambda > \lambda_1$ .

Conversely, if  $h(x) > 0$  on  $\overline{D}$ , then Taira-Umezū [14] proved that problem  $(*)_\lambda$  has a unique positive solution  $u_\lambda(x) \in C^{2+\theta}(\overline{D})$  for each  $\lambda > \lambda_1$ . Furthermore, the solution  $u_\lambda(x)$  grows up as  $\lambda \rightarrow \infty$ , that is, the maximum norm  $\|u_\lambda\|_\infty$  on  $\overline{D}$  tends to infinity as  $\lambda \rightarrow \infty$ .

This paper is concerned with the case where the function  $h(x)$  may vanish in  $D$ . More precisely, we assume that

$$\begin{aligned} &\text{The zero set } D(h) = \{x \in \overline{D} : h(x) = 0\} \text{ of the function} \\ &h(x) \text{ is bounded away from the boundary } \partial D, \end{aligned} \quad (\text{H.4})$$

and denote by  $D_0(h)$  its *interior*. Following del Pino [3], we introduce a critical value  $\lambda_1(D_0(h))$  in the following way: Let  $\mathcal{B}$  be the set of all open subsets of  $D$  with smooth boundary. If  $\Omega \in \mathcal{B}$ , we denote by  $\lambda_1(\Omega)$  the first eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

By the celebrated Rayleigh theorem, we know that the first eigenvalue  $\lambda_1(\Omega)$  is given by the variational formula

$$\lambda_1(\Omega) = \inf \left\{ \int_{\Omega} |\nabla u|^2 dx : u \in H_0^1(\Omega), \int_{\Omega} u^2 dx = 1 \right\},$$

where  $H_0^1(\Omega)$  is the closure of the space  $C_0^\infty(\Omega)$  of smooth functions with compact support in  $\Omega$  in the Sobolev space  $H^1(\Omega)$ . Then we let

$$\lambda_1(D_0(h)) = \sup \{ \lambda_1(\Omega) : \Omega \in \mathcal{B}, D_0(h) \subset \Omega \}. \quad (1.2)$$

We understand  $\lambda_1(D_0(h)) = \infty$  in the case where the set  $D_0(h)$  is empty. Remark that if the boundary  $\partial D_0(h)$  is sufficiently regular, then the value  $\lambda_1(D_0(h))$  coincides with the first eigenvalue of the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta \varphi = \lambda \varphi & \text{in } D_0(h), \\ \varphi = 0 & \text{on } \partial D_0(h). \end{cases}$$

Our first result is the following existence and uniqueness theorem of positive solutions of problem  $(*)_\lambda$ :

**Theorem 1.** *Assume that conditions (H.1) through (H.4) are satisfied. Then problem  $(*)_\lambda$  has a unique positive solution  $u_\lambda(x) \in C^{2+\theta}(\overline{D})$  for every  $\lambda_1 < \lambda < \lambda_1(D_0(h))$  and no positive solution for all  $\lambda \geq \lambda_1(D_0(h))$ . Furthermore the solution  $u_\lambda(x)$  grows up as  $\lambda \uparrow \lambda_1(D_0(h))$ , that is, the maximum norm  $\|u_\lambda\|_\infty$  tends to infinity as  $\lambda \uparrow \lambda_1(D_0(h))$ .*

Theorem 1 is a generalization of del Pino [3, Theorem 2] where the Dirichlet and Neumann conditions are treated, and it is proved by Taira–Umezū [13, Theorem 3] under the condition that the boundary  $\partial D_0(h)$  is sufficiently regular.

Secondly we study the asymptotic behavior of the unique positive solution  $u_\lambda(x)$  as  $\lambda \uparrow \lambda_1(D_0(h))$ . To do so, we take a relatively compact, open subset  $\Omega'$  of  $D$  with smooth boundary  $\partial\Omega'$  which satisfies the conditions

$$\Omega' \supset D(h); \quad (1.3a)$$

The closure  $\overline{\Omega'} = \Omega' \cup \partial\Omega'$  consists of

$$\text{a finite number of connected components.} \quad (1.3b)$$

Then we let

$$\begin{aligned} \Omega &= D \setminus \overline{\Omega'}, \\ \Gamma &= \partial\Omega \cap D, \end{aligned}$$

and introduce a non-negative smooth function  $\rho(x)$  defined on the closure  $\overline{\Omega}$  such that

$$\rho(x) = \begin{cases} \inf\{|x - y| : y \in \Gamma\} & \text{on a tubular neighborhood of the} \\ & \text{topological boundary } \Gamma \text{ of } \Omega \text{ in } D; \\ 1 & \text{on a tubular neighborhood of the} \\ & \text{boundary } \partial D. \end{cases} \quad (1.4)$$

Now we can state our second result which is a generalization of del Pino [3, Theorem 3] to the degenerate case:

**Theorem 2.** *Assume that conditions (H.1) through (H.4) are satisfied. If  $\Omega'$  is a relatively compact, open subset of  $D$  with smooth boundary  $\partial\Omega'$  which satisfies conditions (1.3a) and (1.3b) and if  $\Omega = D \setminus \overline{\Omega'}$ , then, for any bounded sub-interval  $I$  of the interval  $(\lambda_1, \lambda_1(D_0(h)))$  and any  $\alpha > 2/(p-1)$  there exists a constant  $C > 0$  such that we have*

$$\sup_{\lambda \in I} u_\lambda(x) \leq C \rho(x)^{-\alpha}, \quad x \in \Omega. \quad (1.5)$$

*Furthermore, if the interior  $D_0(h)$  is connected and non-empty, then we have, for any compact subset  $K$  of  $D_0(h)$ ,*

$$\inf_{x \in K} u_\lambda(x) \longrightarrow \infty \quad \text{as } \lambda \uparrow \lambda_1(D_0(h)). \quad (1.6)$$

Rephrased, Theorem 2 asserts that the more the exponent  $p$  increases, the milder the solution  $u_\lambda(x)$  behaves; while the more the set  $D_0(h)$  enlarges, the wilder the solution  $u_\lambda(x)$  behaves.

If the set  $D_0(h)$  is equal to the unit open ball in  $\mathbf{R}^N$ , then we can give a precise description of the growing-up rate of the solution  $u_\lambda(x)$  in  $D_0(h)$  (see Theorem 5.1). For a similar description, we refer to del Pino [3, Remark 1] where the growing-up rate in the set  $\overline{D} \setminus D(h)$  is given for a smooth function  $h(x)$ .

Finally, we discuss the behavior of the solution  $u_\lambda(x)$  as  $\lambda \uparrow \lambda_1(D_0(h))$  in the case where the zero set  $D(h)$  is non-empty but its interior  $D_0(h)$  is empty. Then Theorem 1 tells us that there exists a unique positive solution  $u_\lambda(x)$  of problem  $(*)_\lambda$  for each  $\lambda > \lambda_1$  and that the maximum norm  $\|u_\lambda\|_\infty$  tends to infinity as  $\lambda \rightarrow \infty$ .

Our third result generalizes assertion (1.6) of Theorem 2 to the case where the interior  $D_0(h)$  is empty:

**Theorem 3.** *Assume that conditions (H.1) through (H.4) are satisfied, and further that there exists a sequence  $\{\Omega_j\}_{j=1}^\infty$  of relatively compact, open subsets of  $D$  with smooth boundary such that the  $\Omega_j$  contain the zero set  $D(h)$  and satisfy the condition*

$$\lim_{j \rightarrow \infty} |\Omega_j| = 0,$$

*where  $|\cdot|$  denotes the Lebesgue measure of a measurable set of  $\mathbf{R}^N$ . Then problem  $(*)_\lambda$  has a unique positive solution  $u_\lambda(x)$  for each  $\lambda > \lambda_1$  which tends to infinity as  $\lambda \rightarrow \infty$ , uniformly with respect to  $x \in K$  for any compact subset  $K$  of  $D$ .*

**Example 1.** If the zero set  $D(h)$  consists of finitely many points in  $D$ , then Theorem 3 applies.

**Example 2.** If the zero set  $D(h)$  consists of finitely many connected components of dimension  $m$  with  $1 \leq m \leq N-1$ , then Theorem 3 applies.

The rest of this paper is organized as follows.

In Section 2 we prove Theorem 1 by using the super-sub-solution method and comparison arguments with the Dirichlet and Neumann conditions. Section 3 is devoted to the proof of Theorem 2. Our approach is based on a modification of the variational technique of del Pino [3] adapted to the degenerate case. In Section 4

we prove Theorem 3. The essential step in the proof is how to construct a super-solution of problem  $(*)_\lambda$  in order to prove the existence of a positive solution, while we construct a good sub-solution in order to study the behavior of the positive solution, by making use of the eigenfunction  $\varphi_1(x)$  of problem (1.1). In Section 5 we consider the growing-up rate of the unique positive solution  $u_\lambda(x)$  in the case where the interior  $D_0(h)$  is the unit open ball in  $\mathbf{R}^N$  and the function  $h(x)$  satisfies a growth condition near the boundary  $\partial D_0(h)$  (Theorem 5.1). In order to give a precise description of the growing-up rate of the solution  $u_\lambda(x)$ , we transpose problem  $(*)_\lambda$  into an equivalent fixed point equation  $(**)_\lambda$  for the resolvent  $K$ , and then apply the super-sub-solution method (Theorem 5.2).

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## 2. PROOF OF THEOREM 1

In this section we prove Theorem 1 by using the super-sub-solution method and comparison arguments with the Dirichlet and Neumann conditions. By Taira–Umezū [13], we know that the problem  $(*)_\lambda$  has at most one positive solution for every  $\lambda > \lambda_1$ .

(I) First, we prove that problem  $(*)_\lambda$  has a positive solution  $u_\lambda(x)$  for every  $\lambda_1 < \lambda < \lambda_1(D_0(h))$ , by using the super-sub-solution method.

Let  $f(x, t)$  be a real-valued, Hölder continuous function with exponent  $0 < \theta < 1$  on  $\overline{D} \times [0, r]$  for any  $r > 0$ , and satisfy the following *slope condition* or *one-sided Lipschitz condition* (cf. [2], [10]): For any  $r > 0$ , there exists a constant  $L > 0$  such that

$$f(x, t) - f(x, s) > -L(t - s), \quad x \in \overline{D}, \quad 0 \leq s < t \leq r.$$

Now we consider the solvability of the semilinear elliptic problem

$$\begin{cases} -\Delta u = f(x, u) & \text{in } D, \\ Bu = a(x') \frac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))u = 0 & \text{on } \partial D. \end{cases} \quad (2.1)$$

A non-negative function  $\phi(x) \in C^2(\overline{D})$  is called a *sub-solution* of problem (2.1) if it satisfies the conditions

$$\begin{cases} -\Delta \phi \leq f(x, \phi) & \text{in } D, \\ B\phi \leq 0 & \text{on } \partial D. \end{cases}$$

Similarly, a non-negative function  $\psi(x) \in C^2(\overline{D})$  is called a *super-solution* of problem (2.1) if it satisfies the conditions

$$\begin{cases} -\Delta \psi \geq f(x, \psi) & \text{in } D, \\ B\psi \geq 0 & \text{in } \partial D. \end{cases}$$

The next theorem, [13, Theorem 1], plays a fundamental role in the construction of positive solutions of problem (2.1) (cf. [2, Theorem 9.4], [10, Theorems 3.2.1 and 3.2.2] for the non-degenerate case):

**Theorem 2.1.** *Assume that condition (H.3) is satisfied. If there exist a sub-solution  $\phi(x)$  and a super-solution  $\psi(x)$  of problem (2.1) such that  $\phi(x) \leq \psi(x)$  on  $\overline{D}$ , then problem (2.1) has a solution  $u(x) \in C^{2+\theta}(\overline{D})$  such that  $\phi(x) \leq u(x) \leq \psi(x)$  on  $\overline{D}$ .*

(I-a) We construct a super-solution of problem  $(*)_\lambda$  for each  $\lambda_1 < \lambda < \lambda_1(D_0(h))$ , by using the following existence result for the Neumann problem due to del Pino [3] (see Ouyang [8] for the case where  $\partial D_0(h)$  is sufficiently smooth):

**Theorem 2.2.** *Assume that conditions (H.1), (H.2) and (H.4) are satisfied. Then the homogeneous Neumann problem*

$$\begin{cases} -\Delta v = \lambda v - h(x) v^p & \text{in } D, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial D \end{cases} \quad (2.2)$$

*has a unique positive solution  $v_\lambda(x) \in C^{2+\theta}(\overline{D})$  for each  $0 < \lambda < \lambda_1(D_0(h))$ .*

Let  $\psi_\lambda(x)$  be a unique positive solution of problem (2.2) for  $0 < \lambda < \lambda_1(D_0(h))$ . Then it follows that the function  $\psi_\lambda$  is a super-solution of problem  $(*)_\lambda$ , since we have

$$B\psi_\lambda = a(x') \frac{\partial \psi_\lambda}{\partial \mathbf{n}} + (1 - a(x')) \psi_\lambda = (1 - a(x')) \psi_\lambda \geq 0 \quad \text{on } \partial D.$$

(I-b) Next we construct a sub-solution of problem  $(*)_\lambda$ . Let  $\varphi_1(x) \in C^\infty(\overline{D})$  be the positive eigenfunction corresponding to the first eigenvalue  $\lambda_1$  of problem (1.1), normalized as  $\|\varphi_1\|_\infty = 1$ . If  $\lambda > \lambda_1$ , then we have, for  $\varepsilon > 0$ ,

$$-\Delta(\varepsilon\varphi_1) - \lambda\varepsilon\varphi_1 + h(x) (\varepsilon\varphi_1)^p \leq (\|h\|_\infty \varepsilon^{p-1} - (\lambda - \lambda_1)) \varepsilon\varphi_1 \quad \text{in } D.$$

This proves that the function  $\varepsilon_\lambda \varphi_1(x)$  is a sub-solution of problem  $(*)_\lambda$  if  $\varepsilon_\lambda$  is sufficiently small.

(I-c) By [12, Lemma 2.1], we see that the functions  $\psi_\lambda(x)$  and  $\varphi_1(x)$  are comparable. This implies that if  $\varepsilon_\lambda$  is sufficiently small, then it follows that  $\varepsilon_\lambda \varphi_1(x) \leq \psi_\lambda(x)$  on  $\overline{D}$ . Therefore, by applying Theorem 2.1 we can find a positive solution  $u_\lambda(x)$  of problem  $(*)_\lambda$  for every  $\lambda_1 < \lambda < \lambda_1(D_0(h))$  such that

$$\varepsilon_\lambda \varphi_1(x) \leq u_\lambda(x) \leq \psi_\lambda(x) \quad \text{on } \overline{D}.$$

(II) Secondly we prove a non-existence result for all  $\lambda \geq \lambda_1(D_0(h))$ . To do so, we need the following existence and non-existence results for the Dirichlet problem due to del Pino [3] (see Ouyang [8] for the case where  $\partial D_0(h)$  is sufficiently smooth):

**Theorem 2.3.** *Assume that conditions (H.1), (H.2) and (H.4) are satisfied. Then the homogeneous Dirichlet problem*

$$\begin{cases} -\Delta w = \lambda w - h(x) w^p & \text{in } D, \\ w = 0 & \text{on } \partial D \end{cases} \quad (2.3)$$

has a unique positive solution  $w_\lambda(x) \in C^{2+\theta}(\overline{D})$  for each  $\lambda_1(D) < \lambda < \lambda_1(D_0(h))$ , and it has no positive solution for all  $\lambda \geq \lambda_1(D_0(h))$ . Here  $\lambda_1(D)$  is the first eigenvalue of the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

Now assume to the contrary that problem  $(*)_\lambda$  has a positive solution  $u_\lambda(x)$  for some  $\lambda \geq \lambda_1(D_0(h))$ . Then it follows that the function  $u_\lambda(x)$  is a super-solution of problem (2.3), since we have

$$u_\lambda \geq 0 \quad \text{on } \partial D.$$

On the other hand, if  $\varphi_0(x)$  is a positive eigenfunction corresponding to the first eigenvalue  $\lambda_1(D)$ , then it is easy to verify that the function  $\varepsilon_\lambda \varphi_0(x)$  is a sub-solution of problem (2.3) for  $\varepsilon_\lambda$  sufficiently small, since we have  $\lambda > \lambda_1(D)$ .

Therefore, by applying Theorem 2.1 to the Dirichlet case ( $a(x') \equiv 0$  on  $\partial D$ ) we can find a positive solution  $w_\lambda(x) \in C^{2+\theta}(\overline{D})$  of problem (2.3) for  $\lambda \geq \lambda_1(D_0(h))$  such that

$$\varepsilon_\lambda \varphi_0(x) \leq w_\lambda(x) \leq u_\lambda(x) \quad \text{on } \overline{D}.$$

However, this contradicts Theorem 2.3.

(III) Finally, since we have proved the existence and non-existence results for problem  $(*)_\lambda$ , we can prove just as in [13] that the maximum norm  $\|u_\lambda\|_\infty$  tends to infinity as  $\lambda \uparrow \lambda_1(D_0(h))$ .

The proof of Theorem 1 is now complete.  $\square$

### 3. PROOF OF THEOREM 2

This section is devoted to the proof of Theorem 2. Our approach is based on a modification of the variational technique of del Pino [3] adapted to the degenerate case. The proof is divided into four steps.

(I) We introduce a non-negative smooth function  $\rho(x) \in C^\infty(\overline{\Omega})$  defined by formula (1.4), and consider a function

$$v(x) = C \rho(x)^{-\alpha}, \quad \alpha > 2/(p-1),$$

where  $C$  is a positive constant to be chosen later on. Then we have, by a direct computation,

$$-\Delta v = C (\alpha \rho^{-\alpha-1} \Delta \rho - \alpha(\alpha+1) \rho^{-\alpha-2} |\nabla \rho|^2).$$

Since  $\rho(x) = 1$  in a tubular neighborhood of  $\partial D$ , by integration by parts it follows that we have, for all non-negative functions  $\varphi(x) \in C^1(\Omega)$  having support away from  $\Gamma = \partial\Omega \cap D$ ,

$$\begin{aligned} \int_{\Omega} \nabla v \cdot \nabla \varphi \, dx &= - \int_{\Omega} \Delta v \cdot \varphi \, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \mathbf{n}} \varphi \, d\sigma \\ &= C \int_{\Omega} (\alpha \rho^{-\alpha-1} \Delta \rho - \alpha(\alpha+1) \rho^{-\alpha-2} |\nabla \rho|^2) \varphi \, dx, \end{aligned} \quad (3.1)$$



where  $d\sigma$  is the surface element on  $\partial\Omega$ .

On the other hand we find that any positive solution  $u(x)$  of problem  $(*)_\lambda$  satisfies the formula

$$\int_{\Omega} (\nabla u \cdot \nabla \varphi + (h u^p - \lambda u) \varphi) dx = \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} \varphi d\sigma. \quad (3.2)$$

However, since we have

$$Bu = a(x') \frac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))u = 0,$$

it follows that

$$\begin{cases} u(x') = 0 & \text{if } a(x') = 0, \\ \frac{\partial u}{\partial \mathbf{n}}(x') = -\frac{1 - a(x')}{a(x')} u(x') \leq 0 & \text{if } a(x') > 0, \end{cases}$$

so that

$$\frac{\partial u}{\partial \mathbf{n}} \leq 0 \quad \text{on } \partial D.$$

Therefore, combining inequalities (3.1) and (3.2) we have, for all non-negative functions  $\varphi \in C^1(\Omega)$  having support away from  $\Gamma$ ,

$$\begin{aligned} & \int_{\Omega} (\nabla(u - v) \cdot \nabla \varphi + (h u^p - \lambda u) \varphi) dx \\ &= C \int_{\Omega} (-\alpha \rho^{-\alpha-1} \Delta \rho + \alpha(\alpha + 1) \rho^{-\alpha-2} |\nabla \rho|^2) \varphi dx + \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} \varphi d\sigma \\ &\leq C \int_{\Omega} (-\alpha \rho^{-\alpha-1} \Delta \rho + \alpha(\alpha + 1) \rho^{-\alpha-2} |\nabla \rho|^2) \varphi dx. \end{aligned} \quad (3.3)$$

If we let

$$\underline{h} = \frac{1}{2} \inf_{x \in \Omega} h(x),$$

then we obtain from inequality (3.3) that

$$\begin{aligned} & \int_{\Omega} (\nabla(u - v) \cdot \nabla \varphi + (h u^p - \lambda u - \underline{h} v^p) \varphi) dx \\ &\leq C |\Omega| \|\varphi\|_{\infty, \overline{\Omega}} \|\rho\|_{\infty, \overline{\Omega}}^{-\alpha p} \left( \alpha \|\rho\|_{\infty, \overline{\Omega}}^{-\alpha-1+\alpha p} \|\Delta \rho\|_{\infty, \overline{\Omega}} \right. \\ &\quad \left. + \alpha(\alpha + 1) \|\rho\|_{\infty, \overline{\Omega}}^{-\alpha-2+\alpha p} \|\nabla \rho\|_{\infty, \overline{\Omega}} - \underline{h} C^{p-1} \right), \end{aligned} \quad (3.4)$$

where  $\|\cdot\|_{\infty, \overline{\Omega}}$  is the maximum norm of  $C(\overline{\Omega})$ . Since  $\underline{h} > 0$  and  $-\alpha - 2 + \alpha p > 0$ , it follows from inequality (3.4) that

$$\int_{\Omega} (\nabla(u - v) \cdot \nabla \varphi + (h u^p - \lambda u - \underline{h} v^p) \varphi) dx \leq 0, \quad (3.5)$$

if we take the constant  $C$  (independent of  $\lambda$ ) so large that

$$C^{p-1} > \frac{\alpha}{\underline{h}} \left( \|\rho\|_{\infty, \overline{\Omega}} \|\Delta \rho\|_{\infty, \overline{\Omega}} + (\alpha + 1) \|\nabla \rho\|_{\infty, \overline{\Omega}} \right) \|\rho\|_{\infty, \overline{\Omega}}^{-\alpha-2+\alpha p}. \quad (3.6)$$

We remark that formula (3.5) remains valid for all non-negative functions  $\varphi$  in the Sobolev space  $H^1(\Omega)$  having support away from  $\Gamma$ .

(II) Let  $I$  be a bounded sub-interval of the interval  $(\lambda_1, \lambda_1(D_0(h)))$ , and  $u_\lambda(x)$  a positive solution of problem  $(*)_\lambda$  for  $\lambda \in I$ . Then we have the assertion

$$x \in \Omega \text{ and } u_\lambda(x) \geq C \rho(x)^{-\alpha} \implies h(x) u_\lambda(x)^p - \lambda u_\lambda(x) \geq \underline{h} u_\lambda(x)^p, \quad (3.7)$$

if we take a constant  $C > 0$  sufficiently large, independent of  $\lambda \in I$ .

Indeed, if we let

$$\overline{\lambda} = \sup_{\lambda \in I} \lambda,$$

then it follows that

$$\begin{aligned} h(x) u_\lambda(x)^p - \lambda u_\lambda(x) - \underline{h} u_\lambda(x)^p &\geq \underline{h} u_\lambda(x)^p - \overline{\lambda} u_\lambda(x) \\ &= u(x) (\underline{h} u_\lambda(x)^{p-1} - \overline{\lambda}) \\ &\geq u(x) (\underline{h} C^{p-1} \rho(x)^{-\alpha(p-1)} - \overline{\lambda}) \\ &\geq u(x) (\underline{h} C^{p-1} \|\rho\|_{\infty, \overline{\Omega}}^{-\alpha(p-1)} - \overline{\lambda}) \\ &> 0, \end{aligned}$$

if we take the constant  $C > 0$  so large that

$$C^{p-1} > \frac{\overline{\lambda}}{\underline{h}} \|\rho\|_{\infty, \overline{\Omega}}^{\alpha(p-1)}. \quad (3.8)$$

(III) In order to prove assertion (1.5), assume to the contrary that one can find a bounded interval  $I_0 \subset (\lambda_1, \lambda_1(D_0(h)))$  such that, for any constant  $C > 0$  there exist a parameter  $\lambda_0 \in I_0$  and a point  $x_0 \in \Omega$  such that the unique positive solution  $u_{\lambda_0}(x)$  of problem  $(*)_{\lambda_0}$  satisfies the inequality

$$u_{\lambda_0}(x_0) > C \rho(x_0)^{-\alpha}.$$

We choose a large constant  $C$  satisfying inequalities (3.6) and (3.8) with  $I := I_0$ , and let

$$\begin{aligned} u_0(x) &= u_{\lambda_0}(x), \\ v_0(x) &= C \rho(x)^{-\alpha}, \\ w_0(x) &= \max \{u_0(x) - v_0(x), 0\}. \end{aligned}$$

Then it follows that the function  $w_0(x)$  belongs to the Sobolev space  $H^1(\Omega)$  having support away from  $\Gamma$ , since we have, for all  $x$  in a tubular neighborhood of  $\Gamma$  in  $\Omega$ ,

$$u_0(x) < v_0(x).$$

Hence, applying inequality (3.5) to the functions  $u := u_0$ ,  $v := v_0$  and  $\varphi := w_0$  we obtain that

$$\int_{\Omega} (\nabla(u_0 - v_0) \cdot \nabla w_0 + (h u_0^p - \lambda_0 u_0 - \underline{h} v_0^p) w_0) dx \leq 0,$$

or equivalently,

$$\int_{\text{supp } w_0} (\nabla(u_0 - v_0) \cdot \nabla w_0 + (h u_0^p - \lambda_0 u_0 - \underline{h} v_0^p) w_0) dx \leq 0.$$

Furthermore, in view of assertion (3.7) this implies that

$$\int_{\text{supp } w_0} (\nabla(u_0 - v_0) \cdot \nabla w_0 + \underline{h}(u_0^p - v_0^p) w_0) dx \leq 0.$$

Therefore, we conclude that

$$\begin{aligned} 0 &< \int_{\text{supp } w_0} \underline{h}(u_0^p - v_0^p) w_0 dx \\ &\leq \int_{\text{supp } w_0} (\nabla(u_0 - v_0) \cdot \nabla w_0 + \underline{h}(u_0^p - v_0^p) w_0) dx \\ &\leq 0, \end{aligned}$$

since  $w_0 = u_0 - v_0$  on  $\text{supp } w_0$  and the Lebesgue measure of  $\text{supp } w_0$  is positive. This contradiction proves assertion (1.5).

(IV) Finally, it remains to prove assertion (1.6). Let  $u_\lambda(x)$  be a positive solution of problem  $(*)_\lambda$  for any  $\lambda_1(D) < \lambda < \lambda_1(D_0(h))$ , and let  $\varphi_0(x)$  be a positive eigenfunction corresponding to the first eigenvalue  $\lambda_1(D)$  of problem (2.4). Then, just as in step (II) of the proof of Theorem 1 we find that the function  $u_\lambda(x)$  is a super-solution of problem (2.3) and further that the function  $\varepsilon_\lambda \varphi_0(x)$  is a sub-solution of problem (2.3) if  $\varepsilon_\lambda$  is sufficiently small.

Therefore, by applying Theorem 2.1 to the Dirichlet case ( $a(x') \equiv 0$  on  $\partial D$ ) we can find a positive solution  $w_\lambda(x) \in C^{2+\theta}(\overline{D})$  of problem (2.3) such that

$$\varepsilon_\lambda \varphi_0(x) \leq w_\lambda(x) \leq u_\lambda(x), \quad x \in \overline{D}. \quad (3.9)$$

However, we know from part (ii) of [3, Theorem 3] that assertion (1.6) holds for the function  $w_\lambda(x)$ , that is,

$$\inf_{x \in K} w_\lambda(x) \longrightarrow \infty \quad \text{as } \lambda \uparrow \lambda_1(D_0(h)).$$

In view of inequalities (3.9), it follows that assertion (1.6) holds also for the solution  $u_\lambda(x)$  of problem  $(*)_\lambda$ .

Now the proof of Theorem 2 is complete.  $\square$

## 4. PROOF OF THEOREM 3

In this section we prove Theorem 3. The essential step in the proof is how to construct a super-solution of problem  $(*)_\lambda$  in order to prove the existence of a positive solution, while we construct a good sub-solution in order to study the behavior of the positive solution, by making use of the eigenfunction  $\varphi_1(x)$  of problem (1.1).

(I) Now take the positive eigenfunction  $\varphi_1(x)$  corresponding to the first eigenvalue  $\lambda_1$  of problem (1.1) such that  $\|\varphi_1\|_\infty = 1$  and let

$$v(x) = \left( \frac{\lambda - \lambda_1}{\|h\|_\infty} \right)^{1/(p-1)} \varphi_1(x), \quad \lambda > \lambda_1.$$

Then we have

$$-\Delta v - \lambda v + h(x)v^p \leq 0 \quad \text{in } D.$$

This implies that  $v(x)$  is a sub-solution of problem  $(*)_\lambda$ .

(II) Next we construct a super-solution of problem  $(*)_\lambda$ . Since the function  $h(x)$  satisfies conditions (H.1), (H.2) and (H.4), for each  $\lambda > \lambda_1$  we can choose a non-negative function  $\tilde{h}(x) \in C^\theta(\overline{D})$  such that

1. The zero set  $D(\tilde{h})$  of the function  $\tilde{h}(x)$  is bounded away from  $\partial D$ ;
2. The interior  $D_0(\tilde{h})$  of  $D(\tilde{h})$  is not empty;
3. The interior  $D_0(\tilde{h})$  contains the zero set  $D(h)$  of the function  $h(x)$ ;
4.  $\tilde{h}(x) \leq h(x)$ ,  $x \in \overline{D}$ ;
5.  $\lambda < \lambda_1(D_0(\tilde{h}))$ .

Here we have used the fact that the value  $\lambda_1(D_0(h))$  defined by formula (1.2) tends to infinity as the Lebesgue measure  $|D_0(h)|$  goes to zero.

Now we consider the following boundary value problem

$$\begin{cases} -\Delta u = \lambda u - \tilde{h}(x)u^p & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases} \quad (4.1)$$

Theorem 1 tells us that problem (4.1) has a unique positive solution  $w(x) \in C^{2+\theta}(\overline{D})$ . Then it follows that the function  $Cw(x)$  is a super-solution of problem  $(*)_\lambda$  for all  $C \geq 1$ . Indeed, we have

$$-\Delta(Cw) - \lambda Cw + h(x)(Cw)^p = Cw^p(C^{p-1}h(x) - \tilde{h}(x)) \geq 0 \quad \text{in } D,$$

since  $h(x) \geq \tilde{h}(x)$  on  $\overline{D}$  and  $p > 1$ .

(III) Applying Theorem 2.1 to the sub-solution  $v(x)$  and the super-solution  $Cw(x)$  for  $C$  sufficiently large, we can find a solution  $u_\lambda(x)$  of problem  $(*)_\lambda$  such that

$$v(x) = \left( \frac{\lambda - \lambda_1}{\|h\|_\infty} \right)^{1/(p-1)} \varphi_1(x) \leq u_\lambda(x) \leq Cw(x) \quad \text{on } \overline{D}.$$

This proves that the solution  $u_\lambda(x)$  tends to infinity as  $\lambda \rightarrow \infty$ , uniformly with respect to  $x \in K$  for any compact subset  $K$  of  $D$ , since we have  $\varphi_1(x) > 0$  in  $D$ .

The proof of Theorem 3 is now complete.  $\square$

## 5. GROWING-UP RATE OF POSITIVE SOLUTIONS

In this section we study the growing-up rate of the unique positive solution  $u_\lambda(x)$  of problem  $(*)_\lambda$  as  $\lambda \rightarrow \lambda_1(D_0(h))$  under some further restrictions on the function  $h(x)$ , which generalizes assertion (1.6) of Theorem 2 (Theorem 5.1). In order to give a precise description of the growing-up rate of the solution  $u_\lambda(x)$ , we transpose problem  $(*)_\lambda$  into an equivalent fixed point equation  $(**)_\lambda$  for the resolvent  $K$ , and then apply the super-sub-solution method (Theorem 5.2).

We assume that the zero set  $D(h)$  of the function  $h(x)$  is given by the formula

$$D(h) = \{x \in \mathbf{R}^N : |x| \leq 1\}, \quad (5.1)$$

and we let

$$\begin{cases} D_1 := D_0(h) = \{x \in \mathbf{R}^N : |x| < 1\}, \\ D_r := \{x \in \mathbf{R}^N : |x| < r\}, \quad r > 1. \end{cases}$$

Concerning the growth rate of  $h(x)$  near  $\partial D_0(h)$ , we assume that there exist constants  $\sigma > 1$ ,  $C_1 > 0$  and  $\delta_0 > 0$  such that

$$\sup_{x \in D_r} h(x) \leq C_1(r-1)^\sigma, \quad r \in [1, 1 + \delta_0]. \quad (5.2)$$

**Example 5.1.** If  $h(x)$  is a function in  $C^1(\mathbf{R}^N)$  given by the formula

$$h(x) = \begin{cases} 0 & |x| < 1, \\ (|x| - 1)^2 & |x| \geq 1, \end{cases}$$

then  $h(x)$  satisfies conditions (5.1) and (5.2) with  $\sigma = 2$ .

Now let  $\phi_r(x)$  be a positive eigenfunction associated with the first eigenvalue  $\lambda_1(D_r)$  of the Dirichlet problem

$$\begin{cases} -\Delta \varphi = \mu \varphi & \text{in } D_r, \\ \varphi = 0 & \text{on } \partial D_r, \end{cases} \quad (5.3)$$

where the eigenfunction  $\phi_r(x)$  is normalized as  $\|\phi_r\|_{\infty, \overline{D_r}} = 1$ .

The next theorem gives a precise description of the growing-up rate of the positive solution  $u_\lambda(x)$  in assertion (1.6) of Theorem 2:

**Theorem 5.1.** *Assume that the function  $h(x)$  satisfies conditions (5.1) and (5.2) and that its zero set  $D(h)$  is bounded away from the boundary  $\partial D$ . If  $u_\lambda(x)$  is a unique positive solution of problem  $(*)_\lambda$  for  $\lambda_1 < \lambda < \lambda_1(D_0(h))$ , then, for any compact subset  $K$  of  $D_0(h)$  there exists a constant  $C > 0$ , independent of  $\lambda$ , such that*

$$u_\lambda(x) \geq C (\lambda_1(D_0(h)) - \lambda)^{(1-\sigma)/(p-1)} \phi_1(x) \quad \text{for all } x \in K, \quad (5.4)$$

for  $\lambda$  sufficiently close to  $\lambda_1(D_0(h))$ .

*Proof.* The proof is divided into four steps.

(I) First, we transpose problem  $(*)_\lambda$  into an equivalent fixed point equation for the resolvent of the linearized boundary value problem. By using [13, Theorem 1.1], for a given constant  $d > 0$  we can associate with the boundary value problem

$$\begin{cases} (-\Delta + d)u = f & \text{in } D, \\ Bu = 0 & \text{on } \partial D, \end{cases} \quad (5.5)$$

a linear operator

$$K_d : C^\theta(\overline{D}) \longrightarrow C^{2+\theta}(\overline{D})$$

in the following way: For any function  $f \in C^\theta(\overline{D})$ , the function  $u = K_d f \in C^{2+\theta}(\overline{D})$  is the unique solution of problem (5.5). Then it is easy to verify that the operator  $K$  is uniquely extended to a strictly positive, compact linear operator  $K$  from the ordered Banach space  $C(\overline{D})$  into itself ([12, Lemma 2.1]). Furthermore, we find that problem  $(*)_\lambda$  is equivalent to a nonlinear operator equation

$$u = K_d((\lambda + d)u - hu^p) \quad \text{in } C(\overline{D}). \quad (**)_\lambda$$

(II) Secondly, we apply the super-sub-solution method to solve equation  $(**)_\lambda$ .

A non-negative function  $\phi(x) \in C(\overline{D})$  is said to be a *super-solution* of equation  $(**)_\lambda$  if it satisfies the condition

$$\phi(x) \geq K_d((\lambda + d)\phi - h\phi^p)(x) \quad \text{for } x \in \overline{D}.$$

Similarly, a non-negative function  $\psi(x) \in C(\overline{D})$  is said to be a *sub-solution* of equation  $(**)_\lambda$  if it satisfies the condition

$$\psi(x) \leq K_d((\lambda + d)\psi - h\psi^p)(x) \quad \text{for } x \in \overline{D}.$$

The next existence theorem for problem  $(*)_\lambda$  is implicitly proved in the proof of [13, Theorem 1]:

**Theorem 5.2.** *Let  $\phi(x)$  and  $\psi(x)$  be respectively a sub-solution and a super-solution of equation  $(**)_\lambda$  such that  $\phi(x) \leq \psi(x)$  on  $\overline{D}$ . If the function*

$$g_d(x, t) = (\lambda + d)t - h(x)t^p$$

*is monotonically increasing in  $t$ , that is, if we have*

$$g_d(x, s) < g_d(x, t) \quad \text{for all } x \in \overline{D} \text{ and } 0 \leq s < t \leq \|\psi\|_\infty,$$

*then equation  $(**)_\lambda$  has a fixed point  $u(x) \in C(\overline{D})$  such that*

$$\phi(x) \leq u(x) \leq \psi(x) \quad \text{on } \overline{D}.$$

*In this case, the function  $u(x)$  is a solution of problem  $(*)_\lambda$  in the space  $C^{2+\theta}(\overline{D})$ .*

(III) We construct a sub-solution of equation  $(**)_\lambda$ . To do so, we need the following elementary results:

**Lemma 5.1.** *If  $r_1 > r_2 \geq 1$ , then the first eigenvalue  $\lambda_1(D_r)$  of problem (5.3) and its associated eigenfunction  $\phi_r(x)$  satisfy respectively the conditions*

$$\lambda_1(D_{r_1}) = \left(\frac{r_2}{r_1}\right)^2 \lambda_1(D_{r_2}), \quad (5.6)$$

$$\phi_{r_1}(x) = \phi_{r_2}\left(\frac{r_2}{r_1}x\right), \quad x \in D_{r_1}. \quad (5.7)$$

Since the eigenvalue  $\lambda_1(D_r)$  depends continuously on  $r$ , it follows that, for each  $\lambda < \lambda_1(D_0(h))$  close to  $\lambda_1(D_0(h))$  there exists a constant  $r > 1$  such that

$$\lambda = \lambda_1(D_r).$$

If we let

$$\lambda' = \lambda_1(D_{2r-1}),$$

then we obtain that its associated eigenfunction  $\phi_{2r-1}(x)$  satisfies the conditions

$$\begin{cases} (-\Delta + d)(\varepsilon\phi_{2r-1}) \leq (\lambda + d)\varepsilon\phi_{2r-1} - h(x)(\varepsilon\phi_{2r-1})^p & \text{in } D_{2r-1}, \\ \varepsilon\phi_{2r-1} = 0 & \text{on } \partial D_{2r-1}, \end{cases} \quad (5.8)$$

if  $\varepsilon$  may be chosen to be so small that

$$0 < \varepsilon \leq \left(\frac{\lambda - \lambda'}{\sup_{D_{2r-1}} h}\right)^{1/(p-1)}. \quad (5.9)$$

However, by using condition (5.2) and formula (5.6) we can prove that

$$\left(\frac{\lambda - \lambda'}{\sup_{D_{2r-1}} h}\right)^{1/(p-1)} \geq \left\{\frac{\lambda_1(D_r)(3r-1)}{2^\sigma C_1(2r-1)^2}\right\}^{1/(p-1)} (r-1)^{(1-\sigma)/(p-1)}.$$

This implies that there exists a constant  $\tilde{C} > 0$ , independent of  $r$  close to 1, such that condition (5.9) is valid for

$$\varepsilon = \tilde{C}(r-1)^{(1-\sigma)/(p-1)}.$$

Now we define a continuous function  $v_r(x) \in C(\overline{D})$  as

$$v_r(x) = \begin{cases} \tilde{C}(r-1)^{(1-\sigma)/(p-1)}\phi_{2r-1}(x) & \text{in } D_{2r-1}, \\ 0 & \text{on } \overline{D} \setminus D_{2r-1}. \end{cases}$$

Then, by assertion (5.8) it is easy to see that the function  $v_r(x)$  is a sub-solution of equation  $(**)_{\lambda}$ . In fact, we have the following:

**Lemma 5.2.** *There exists a constant  $d_1 > 0$  such that, for all  $d > d_1$  the function  $v_r(x)$  satisfies the condition*

$$v_r(x) \leq K_d((\lambda + d)v_r - hv_r^p) \quad \text{on } \overline{D}.$$

(IV) *End of Proof of Theorem 5.1.* First, by [5, Theorem 3.2] it follows that there exists a super-solution  $w_\lambda(x) \in C^2(\overline{D})$  of problem  $(*)_\lambda$  for each  $\lambda_1 < \lambda < \lambda_1(D_0(h))$ . Then we remark that the functions  $Rw_\lambda(x)$  are super-solutions of equation  $(**)_\lambda$  for all  $R > 1$ . Moreover we can choose constants  $R_0 > 1$  and  $d > d_1$  so large that

$$v_r(x) \leq R_0 w_\lambda(x) \quad \text{on } \overline{D}$$

and

$$g_d(x, s) < g_d(x, t) \quad \text{for all } x \in \overline{D} \text{ and } 0 \leq s < t < R_0 \|w_\lambda\|_\infty.$$

Hence it follows from an application of Theorem 5.2 that problem  $(*)_\lambda$  has a solution  $u(x) \in C^{2+\theta}(\overline{D})$  such that

$$v_r(x) \leq u(x) \leq R_0 w_\lambda(x) \quad \text{on } \overline{D}.$$

However, by the uniqueness theorem for problem  $(*)_\lambda$  (Theorem 1) we obtain that  $u(x) = u_\lambda(x)$  in  $D$ , so that

$$v_r(x) = \tilde{C}(r-1)^{(1-\sigma)/(p-1)} \phi_{2r-1}(x) \leq u_\lambda(x), \quad x \in D_{2r-1}. \quad (5.10)$$

Furthermore, we have, by formula (5.6),

$$\begin{aligned} r-1 &= \frac{(2r-1)^2}{3r-1} \left( \frac{\lambda_1(D_r) - \lambda_1(D_{2r-1})}{\lambda_1(D_r)} \right) \\ &= \frac{r^2(2r-1)}{r+1} \left( \frac{\lambda_1(D_0(h)) - \lambda}{\lambda_1(D_0(h))} \right). \end{aligned}$$

Summing up, we can rewrite inequality (5.10) in the form

$$C(\lambda_1(D_0(h)) - \lambda)^{(1-\sigma)/(p-1)} \phi_{2r-1}(x) \leq u_\lambda(x), \quad x \in D_{2r-1}, \quad (5.11)$$

where  $C$  is a positive constant independent of  $\lambda$ .

On the other hand, by formula (5.7) it follows that, as  $r \downarrow 1$

$$\phi_{2r-1}(x) \longrightarrow \phi_1(x) \quad \text{in } C(\overline{D_0(h)}). \quad (5.12)$$

Therefore, the desired assertion (5.4) follows by combining inequality (5.11) and assertion (5.12).

Now the proof of Theorem 5.1 is complete.  $\square$



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